Assignment 3.

Möbius transformations. Complex differentiation

This assignment is due Wednesday, Feb 6. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Find Möbius transformation that carries points $-1, i, 1+i$ into
 (a) $0, 2i, 1-i$, (*Hint:* From $-1 \to 0$ you know that it has the form $\frac{z+1}{az+b}$.)
 (b) $i, \infty, 1$. (*Hint:* As above, note that $i \to \infty$.)

(2) (a) Let $f(z) = \frac{az+b}{cz+d}$ and $g(z) = \frac{a'z+b'}{c'z+d'}$ be two Möbius transformations. Compute directly $f \circ g$.
 (b) Compare your result above to matrix multiplication $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.
 (c) Find the condition for a, b, c, d to define a Möbius transformation equal to the identity map $f(z) = z$.

 Remark. Results of Problem 2 above can be formulated in a sciency way: Consider the set of invertible complex 2×2 matrices, denoted

$$GL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\}.$$

 Let φ be the map defined by

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az+b}{cz+d}.$$

 By 2ab, φ is a group homorphism with kernel aE found in 2c, so the group of Möbius transformations is isomorphic to the quotient group $GL(2, \mathbb{C})/aE$. The latter is denotedy by $PSL(2, \mathbb{C})$ and called *projective special linear group.*

(3) Find the images of the following domains under the indicated Möbius transformations:
 (a) The quadrant $x > 0$, $y > 0$ if $w = \frac{z-i}{z+i}$.
 (b) The half-disc $|z| < 1$, $\operatorname{Im} z > 0$ if $w = \frac{2z-i}{2+iz}$.
 (c) The strip $0 < x < 1$ if $w = \frac{z}{z-1}$.
 (d) The strip $0 < x < 1$ if $w = \frac{z-1}{z-2}$.
 (*Hint:* You mainly need to keep track of the borders. It should help a bit to keep in mind that under $z \to \frac{1}{z-b} + b$, straight lines that do not pass through b go to circles that pass through b; and straight lines that pass through b, go to straight lines that pass through b.)

(4) Show that the function $f(z) = z\operatorname{Re} z$ is differentiable only at the point $z = 0$, and find $f'(0)$.

(5) Find $v(x, y)$ such that the function $f(z) = 2xy + iv(x, y)$ is complex differentiable. Express f as a function of z.

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(6) Show that in polar coordinates, at every nonzero point of $\mathbb{C}$, Cachy–Riemann equations take form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \varphi}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \varphi}.$$

(*Hint:* Differentiate $\frac{\partial u(x,y)}{\partial r} = \frac{\partial u(r\cos\varphi, r\sin\varphi)}{\partial r}$ using chain rule. Do the same with other partial derivations $\frac{\partial u}{\partial \varphi}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \varphi}$. Use "usual" Cauchy–Riemann equations.)

(7) Let $z_0 \neq 0$ and let $f(z) = \ln r + i\varphi$, where $r = |z|$, $\varphi \in \operatorname{Arg} z$, and φ is chosen so that f is continuous in a neighborhood of z_0. Prove that f is differentiable in a neighborhood of z_0.

(8) Find an angle by which tangents to curves at z_0 are rotated under the mapping $w = z^2$ if
(a) $z_0 = i$, (b) $z_0 = -1/4$, (c) $z_0 = 1 + i$.
Also find the corresponding values of magnification.

(9) Which part of the plane is shrunk and which part stretched under the following maps: (a) $w = z^2$, (b) $w = z^2 + 2z$, (c) $w = 1/z$? (*Hint:* Whether a map f shrinks or stretches at z_0 depends on $|f'(z_0)|$.)